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Signless Laplacian spectral characterization of the cones over some regular graphs

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ABSTRACT

Let G be an r-regular graph of order n. We prove that the cone over G is determined by its signless Laplacian spectrum for $r = 1, n-2$, for $r = 2$ and $n \geqslant 11$. For $r = n-3$, we show that the cone over G is determined by its signless Laplacian spectrum if and only if the complement of G has no triangles. A class of Q-cospectral graphs are also given.

1. Introduction

Let G be a simple, undirected graph. Let $A(G)$ be the adjacency matrix of G, $D(G)$ be the diagonal matrix of vertex degrees of G. The matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* of G. The spectrum of $A(G)$ and $Q(G)$ are called the *adjacency spectrum* and the *signless Laplacian spectrum* of G, respectively. The eigenvalues of $A(G)$ and $Q(G)$ are called the *A-eigenvalues* and the *Q-eigenvalues* of G, respectively. We use $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ and $q_1(G) \geqslant q_2(G) \geqslant \cdots \geqslant q_n(G)$ to denote the A-eigenvalues and the Q-eigenvalues of G, respectively. A graph is said to *be determined by its adjacency spectrum* (resp. *signless Laplacian spectrum*) if there is no other non-isomorphic graph with the same adjacency spectrum (resp. signless Laplacian spectrum). Two graphs are said to be *Q-cospectral* if they have the same signless Laplacian spectrum.

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For two disjoint graphs G and H, let $G \cup H$ denote the *disjoint union* of G and H, while mG denotes the disjoint union of m copies of G. Let $\overline{G}$ denote the complement of G. The *product* of G and H, denoted by $G \times H$, is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. Clearly $\overline{G \times H} = \overline{G} \cup \overline{H}$. Let K_n, C_n and $K_{1,n-1}$ denote the complete graph, the cycle and the star of order n, respectively. Specially, K_1 stands for an isolated vertex. The graph $G \times K_1$ is called the *cone* over G. Let $\phi_A(G, x)$ (resp. $\phi_Q(G, x)$) denote the characteristic polynomial of the adjacency matrix (resp. signless Laplacian matrix) of G.

Up to now, only some graphs with special structures are shown to be determined by their spectra (see [1,5–10]). In this paper, we mainly investigate the signless Laplacian spectral characterization of the cones over some regular graphs. For an r-regular graph G of order n, we prove that $G \times K_1$ is determined by its signless Laplacian spectrum for $r = 1, n-2$, for $r = 2$ and $n \geqslant 11$. For $r = n-3$, we show that $G \times K_1$ is determined by its signless Laplacian spectrum if and only if $\overline{G}$ has no triangles. We also give a class of Q-cospectral graphs.

2. Some lemmas

In order to get our main results in this paper, we give some helpful lemmas in this section.

Lemma 2.1 [2,5]. *The A-eigenvalues of cycle C_n is $2\cos\frac{2\pi j}{n}$ ($j = 1, 2, \ldots, n$).*

Lemma 2.2 [2]. *Let G be a graph with n vertices, m edges, t triangles and vertex degrees $d_1, d_2, \ldots, d_n$. Let $T_k = \sum_{i=1}^{n}(q_i(G))^k$, then*

$$T_0 = n, \quad T_1 = \sum_{i=1}^{n} d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6t + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

Lemma 2.3 [4]. *Let G be a graph with maximum degree d_1 and second maximum degree d_2. Then*

$$q_2(G) \geqslant d_2 - 1.$$

If $q_2(G) = d_2 - 1$, then $d_1 = d_2$.

Lemma 2.4 [2]. *Let G be a graph on n vertices with vertex degrees $d_1, d_2, \ldots, d_n$. Then*

$$\min\{d_i + d_j\} \leqslant q_1(G) \leqslant \max\{d_i + d_j\},$$

where (i, j) runs over all pairs of adjacent vertices of G.

Lemma 2.5 [4]. *Let G be a connected graph of order $n(n > 1)$, and the minimum degree of G is δ. Then $q_n(G) < \delta$.*

Lemma 2.6 [3]. *Let G be an r-regular graph of order n, then G is determined by its adjacency spectrum when $r = 0, 1, 2, n-3, n-2, n-1$.*

Lemma 2.7 [2]. *Let G be an r-regular graph of order n. Then*

$$\phi_A(\overline{G}, x) = (-1)^n \frac{x - n + r + 1}{x + r + 1}\phi_A(G, -x-1),$$

i.e., the adjacency spectrum of $\overline{G}$ is

$$n - r - 1, -\lambda_2(G) - 1, -\lambda_3(G) - 1, \ldots, -\lambda_n(G) - 1.$$

3. Main results

Let $J_{m\times n}$ denote the $m \times n$ all-one matrix. As usual, I_n stands for the identity matrix of order n. It is well known that the largest A-eigenvalue of an r-regular graph is r.

Theorem 3.1. *Let G be an r-regular graph of order n. Then*

$$\phi_Q(G\times K_1, x) = \frac{x^2-(n+2r+1)x+2nr}{x-2r-1}\phi_A(G, x-r-1),$$

i.e., the signless Laplacian spectrum of $G \times K_1$ is

$$\frac{n+2r+1\pm\sqrt{(n-2r+1)^2+8r}}{2}, \lambda_2(G)+r+1, \ldots, \lambda_n(G)+r+1.$$

Proof. Let $A(G\times K_1)$ be the adjacency matrix of $G\times K_1$, $D(G\times K_1)$ be the diagonal matrix of vertex degrees of $G\times K_1$, then

$$A(G\times K_1) = \begin{pmatrix} 0 & J_{1\times n} \\ J_{n\times 1} & A(G) \end{pmatrix}, \quad D(G\times K_1) = \mathrm{diag}(n, r+1, r+1, \ldots, r+1),$$

$$\phi_Q(G\times K_1, x) = |xI_{n+1} - D(G\times K_1) - A(G\times K_1)| = |\mu I_{n+1} - B|,$$

where $A(G)$ is the adjacency matrix of G, $\mu = x-r-1$, $B = A(G\times K_1)+\mathrm{diag}(n-r-1, 0, 0, \ldots, 0)$. By calculating we get

$$B = \begin{pmatrix} n-r-1 & J_{1\times n} \\ J_{n\times 1} & A(G) \end{pmatrix},$$

$$|\mu I_{n+1} - B| = \begin{vmatrix} \mu-(n-r-1) & -J_{1\times n} \\ -J_{n\times 1} & \mu I_n - A(G) \end{vmatrix}.$$

Let $M_{1,1}, M_{1,2}, \ldots, M_{1,n+1}$ be the cofactor of the entries in the first row of $|\mu I_{n+1} - B|$, then

$$\phi_Q(G\times K_1, x) = [\mu-(n-r-1)]M_{1,1} + M_{1,2} - \cdots + (-1)^{n+1}M_{1,n+1}.$$

Since G is an r-regular graph, every row sum of $\mu I_n - A(G)$ is $\mu - r$. So we have

$$\begin{aligned} \phi_A(G, \mu) &= |\mu I_n - A(G)| = M_{1,1} = -(\mu-r)M_{1,2} = \cdots = (-1)^n(\mu-r)M_{1,n+1}, \\ &-\frac{1}{\mu-r}M_{1,1} = M_{1,2} = -M_{1,3} = \cdots = (-1)^{n+1}M_{1,n+1}, \end{aligned}$$

$$\phi_Q(G\times K_1, x) = [\mu-(n-r-1)]M_{1,1} + nM_{1,2} = \left[\mu-(n-r-1)-\frac{n}{\mu-r}\right]M_{1,1}.$$

By $\mu = x-r-1$ and $M_{1,1} = |\mu I_n - A(G)| = \phi_A(G, \mu)$, we get

$$\phi_Q(G\times K_1, x) = \frac{x^2-(n+2r+1)x+2nr}{x-2r-1}\phi_A(G, x-r-1).$$

Since $\lambda_1(G) = r$, the signless Laplacian spectrum of $G \times K_1$ is

$$\frac{n+2r+1 \pm \sqrt{(n-2r+1)^2+8r}}{2}, \lambda_2(G)+r+1, \ldots, \lambda_n(G)+r+1. \quad \square$$

Lemma 3.2. *Let G be an r-regular graph of order n. Let H be a graph Q-cospectral with $G \times K_1$, and the maximum degree of H is n. Then $H = G \times K_1$ when $r = 0, 1, 2, n-3, n-2, n-1$.*

Proof. Since H is Q-cospectral with $G \times K_1$, H has $n+1$ vertices. Suppose $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{n+1}$ are the vertex degrees of H. By Lemma 2.2 we get

$$\sum_{i=1}^{n+1} d_i = n + (r+1)n, \quad \sum_{i=1}^{n+1} d_i^2 = n^2 + (r+1)^2 n.$$

By $d_1 = n$ we have

$$\sum_{i=2}^{n+1} d_i = (r+1)n, \quad \sum_{i=2}^{n+1} d_i^2 = (r+1)^2 n, \quad \sum_{i=2}^{n+1} (d_i - r - 1)^2 = 0,$$

$$d_2 = d_3 = \cdots = d_{n+1} = r+1.$$

Considering the vertex degrees of H, we have $H = G_1 \times K_1$, where G_1 is an r-regular graph of order n. By Theorem 3.1 we know that G and G_1 have the same adjacency spectrum. If $r = 0, 1, 2, n-3, n-2, n-1$, by Lemma 2.6, we have $G_1 = G$, $H = G_1 \times K_1 = G \times K_1$. $\square$

Theorem 3.3. *Let G be a 1-regular graph of order n, then $G \times K_1$ is determined by its signless Laplacian spectrum.*

Proof. If $n = 2$, then $G = K_2$. Clearly $G \times K_1 = C_3$ is determined by its signless Laplacian spectrum. So we can assume that $n > 2$. Suppose $G = mK_2$, where $m > 1$. Then $n = 2m$ is even. Since the adjacency spectrum of G is $(1)^m, (-1)^m$, by Theorem 3.1, the signless Laplacian spectrum of $G \times K_1$ is

$$\frac{n+3 \pm \sqrt{(n-1)^2+8}}{2}, (3)^{m-1}, (1)^m.$$

Let H be any graph Q-cospectral with $G \times K_1$. Then

$$q_1(H) = \frac{n+3+\sqrt{(n-1)^2+8}}{2} > n+1, \quad q_2(H) = 3, \quad q_{n+1}(H) = 1.$$

Suppose $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{n+1}$ are the vertex degrees of H. From Lemma 2.2 we have

$$\sum_{i=1}^{n+1} d_i = n + 2n = 3n, \quad \sum_{i=1}^{n+1} d_i^2 = n^2 + 4n.$$

From Lemma 2.3 we get $d_2 \leqslant q_2(H) + 1 = 4$. If $d_{n+1} = 0$, then $q_{n+1}(H) = 0$, a contradiction to $q_{n+1}(H) = 1$. If $d_{n+1} = 1$, then there exist a component H_1 of H such that the minimum degree of H_1 is 1. By Lemma 2.5 we have $q_{n+1}(H) < 1$, a contradiction to $q_{n+1}(H) = 1$. So we have $d_{n+1} \geqslant 2$. Since H has $n+1$ vertices, $d_1 \leqslant n$. If $d_1 = n$, by Lemma 3.2, we get $H = G \times K_1$. If $d_2 \leqslant 2$, by $\sum_{i=1}^{n+1} d_i = 3n$, we have $d_1 \geqslant n$, $d_1 = n$, $H = G \times K_1$. Next we only consider the case that $3 \leqslant d_2 \leqslant 4$, $d_{n+1} \geqslant 2$.

Suppose that there are a_3 three and a_2 two in $d_2, d_3, \ldots, d_{n+1}$. By $\sum_{i=1}^{n+1} d_i = 3n$ and $\sum_{i=1}^{n+1} d_i^2 = n^2 + 4n$, we have

$$\begin{cases} a_2 + a_3 + a_4 = n \\ 2a_2 + 3a_3 + 4a_4 = 3n - d_1 \\ 4a_2 + 9a_3 + 16a_4 = n^2 + 4n - d_1^2. \end{cases} \tag{1}$$

Solving Eq. (1), we get $a_3 = (n - d_1)(6 - n - d_1)$. Since $a_3 \geqslant 0$ and $n = 2m \geqslant 4$, $d_1 \leqslant 2$, a contradiction to $d_2 \geqslant 3$. $\square$

Theorem 3.4. *Let G be a $(n-2)$-regular graph of order n, then $G \times K_1$ is determined by its signless Laplacian spectrum.*

Proof. The complement of G is the disjoint union of several K_2. Assume that $\overline{G} = mK_2$. Then the adjacency spectrum of $\overline{G}$ is $(1)^m, (-1)^m$. By Lemma 2.7, the adjacency spectrum of G is $n-2, (0)^m, (-2)^{m-1}$. Theorem 3.1 implies that the signless Laplacian spectrum of $G \times K_1$ is

$$\frac{3n - 3 \pm \sqrt{(n-1)^2 + 8}}{2}, (n-3)^{m-1}, (n-1)^m.$$

Let H be any graph Q-cospectral with $G \times K_1$. Then

$$q_1(H) = \frac{3n - 3 + \sqrt{(n-1)^2 + 8}}{2} > 2n - 2.$$

Let d_1 be the maximum degree of H. Since H has $n+1$ vertices, $d_1 \leqslant n$. From Lemma 2.4 we get $2d_1 \geqslant q_1(H) > 2n - 2$, $d_1 > n - 1$, $d_1 = n$. From Lemma 3.2 we can get $H = G \times K_1$. $\square$

Theorem 3.5. *Let G be a 2-regular graph of order n, and $n \geqslant 11$. Then $G \times K_1$ is determined by its signless Laplacian spectrum.*

Proof. Suppose $G = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_j}$, where $j \geqslant 1$, $n_1 + n_2 + \cdots + n_j = n \geqslant 11$. Let H be any graph Q-cospectral with $G \times K_1$. By Lemma 2.1 and Theorem 3.1, the signless Laplacian spectrum of H is

$$\frac{n + 5 \pm \sqrt{(n-3)^2 + 16}}{2}, \ (5)^{j-1},$$

$$3 + 2\cos\frac{2\pi i}{n_1} (i = 1, 2, \ldots, n_1 - 1), \ldots, 3 + 2\cos\frac{2\pi i}{n_j} (i = 1, 2, \ldots, n_j - 1).$$

Then we have

$$q_1(H) = \frac{n + 5 + \sqrt{(n-3)^2 + 16}}{2} > n + 1, \quad q_2(H) \leqslant 5, \quad q_{n+1}(H) \geqslant 1.$$

Suppose $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_{n+1}$ are the vertex degrees of H. From Lemma 2.2 we have

$$\sum_{i=1}^{n+1} d_i = n + 3n = 4n, \quad \sum_{i=1}^{n+1} d_i^2 = n^2 + 9n.$$

From Lemma 2.3 and Lemma 2.4 we can get

$$d_2 \leqslant q_2(H) + 1 \leqslant 6, \quad d_1 + d_2 \geqslant q_1(H) > n + 1.$$

If $d_{n+1} = 0$, then $q_{n+1}(H) = 0$, a contradiction to $q_{n+1}(H) \geqslant 1$. If $d_{n+1} = 1$, Lemma 2.5 implies that $q_{n+1}(H) < 1$, a contradiction. So we have $d_{n+1} \geqslant 2$.

Since H has $n+1$ vertices, $d_1 \leqslant n$. If $d_1 = n$, by Lemma 3.3, we get $H = G \times K_1$. If $d_2 \leqslant 3$, by $\sum_{i=1}^{n+1} d_i = 4n$ we get $d_1 \geqslant n$, $d_1 = n$, $H = G \times K_1$. If $d_2 = q_2(H) + 1 = 6$, by Lemma 2.3, we have $d_1 = d_2 = 6$. By $d_1 + d_2 > n + 1$, we get $n < 11$, a contradiction to $n \geqslant 11$. Next we only consider the case that $d_{n+1} \geqslant 2,\ 4 \leqslant d_2 \leqslant 5, d_1 \leqslant n - 1$. By $d_1 + d_2 > n + 1$, we get $d_1 \geqslant n - 3$.

Suppose that there are a_5 five, a_4 four, a_3 three and a_2 two in $d_2, d_3, \ldots, d_{n+1}$. By $\sum_{i=1}^{n+1} d_i = 4n$ and $\sum_{i=1}^{n+1} d_i^2 = n^2 + 9n$, we get

$$\begin{cases} a_2 + a_3 + a_4 + a_5 = n \\ 2a_2 + 3a_3 + 4a_4 + 5a_5 = 4n - d_1 \\ 4a_2 + 9a_3 + 16a_4 + 25a_5 = n^2 + 9n - d_1^2. \end{cases} \tag{2}$$

Solving Eq. (2), we get

$$\begin{aligned} a_2 &= \frac{(n-d_1)(n+d_1-5)}{2} + d_1 - n - a_5, \\ a_3 &= 3a_5 + 2n - d_1 - (n-d_1)(n+d_1-5), \\ a_4 &= \frac{(n-d_1)(n+d_1-5)}{2} - 3a_5. \end{aligned}$$

By $a_3 \geqslant 0$ and $a_4 \geqslant 0$ we have

$$(n-d_1)(n+d_1-5) + d_1 - 2n \leqslant 3a_5 \leqslant \frac{(n-d_1)(n+d_1-5)}{2},$$

$$\frac{(n-d_1)(n+d_1-5)}{2} + d_1 - 2n \leqslant 0.$$

Recall that $n-3 \leqslant d_1 \leqslant n-1$ and $n \geqslant 11$. If $d_1 = n-3$ or $n-2$, then $\frac{(n-d_1)(n+d_1-5)}{2} + d_1 - 2n > 0$. Hence $d_1 = n-1$. In this case, we have $a_2 = n - 4 - a_5$, $a_3 = 3a_5 + 7 - n$, $a_4 = n - 3 - 3a_5$. By $a_4 \geqslant 0$ we get $a_5 \leqslant \frac{n-3}{3}$. Since G is a 2-regular graph of order n, $G \times P_1$ has at least n triangles. Assume that H has t triangles. By Lemma 2.2 we have

$$6t + (n-1)^3 + 8a_2 + 27a_3 + 64a_4 + 125a_5 \geqslant 6n + n^3 + 27n,$$

$$6t \geqslant 3n^2 - 15n + 36 - 6a_5.$$

Since there are three $K_{1,2}$ in all subgraphs of a triangle C_3, the number of $K_{1,2}$ in all subgraphs of H is larger than or equal to $3t$. Let m be the number of $K_{1,2}$ in all subgraphs of H, then

$$m = \sum_{i=1}^{n+1} \frac{d_i(d_i-1)}{2} = \sum_{i=1}^{n+1} \frac{d_i^2}{2} - \sum_{i=1}^{n+1} \frac{d_i}{2} = \frac{n^2+9n}{2} - \frac{4n}{2} = \frac{n^2+5n}{2} \geqslant 3t,$$

$$n^2 + 5n \geqslant 6t \geqslant 3n^2 - 15n + 36 - 6a_5,$$

$$n^2 - 10n + 18 - 3a_5 \leqslant 0.$$

By $a_5 \leqslant \frac{n-3}{3}$, we get $n^2 - 11n + 21 \leqslant n^2 - 10n + 18 - 3a_5 \leqslant 0$. By $n \geqslant 11$ we get $n^2 - 11n + 21 > 0$, a contradiction. $\square$

Theorem 3.6. *Let H be any graph of order n. Then $K_{1,3} \times H$ and $(C_3 \cup K_1) \times H$ are Q-cospectral.*

Proof. The signless Laplacian matrix of $(C_3 \cup K_1) \times H$ is

$$M = \begin{pmatrix} Q_1 + nI_4 & J_{4\times n} \\ J_{n\times 4} & Q(H) + 4I_n \end{pmatrix},$$

where $Q_1 = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is the signless Laplacian matrix of $C_3 \cup K_1$, $Q(H)$ is the signless Laplacian matrix of H. Let $P = \begin{pmatrix} \frac{1}{2}J_{4\times 4} - I_4 & 0 \\ 0 & I_n \end{pmatrix}$, then $P^{-1} = P$ and $PMP^{-1} = \begin{pmatrix} \left(\frac{1}{2}J_{4\times 4} - I_4\right) Q_1 \left(\frac{1}{2}J_{4\times 4} - I_4\right) + nI_4 & J_{4\times n} \\ J_{n\times 4} & Q(H) + 4I_n \end{pmatrix}$.

By straightforward computation, we have $\left(\frac{1}{2}J_{4\times 4} - I_4\right) Q_1 \left(\frac{1}{2}J_{4\times 4} - I_4\right) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$, this is just the signless Laplacian matrix of the star $K_{1,3}$. Since PMP^{-1} is the signless Laplacian matrix of $K_{1,3} \times H$, $K_{1,3} \times H$ and $(C_3 \cup P_1) \times H$ are Q-cospectral. $\square$

Let $N_F(G)$ denote the number of subgraphs of G isomorphic to a graph F. Let $t(G)$ denote the number of triangles in G.

Theorem 3.7. *Let G be a $(n-3)$-regular graph of order n. Then $G \times K_1$ is determined by its signless Laplacian spectrum if and only if $\overline{G}$ has no triangles.*

Proof. Clearly $\overline{G}$ is a 2-regular graph of order n. Lemma 2.1 implies that $|\lambda_i(\overline{G})| \leqslant 2$ $(i = 1, 2, \ldots, n)$. By Lemma 2.7 and Theorem 3.1, the signless Laplacian spectrum of $G \times K_1$ is

$$\frac{3n - 5 \pm \sqrt{(n-3)^2 + 16}}{2}, n - 3 - \lambda_2(\overline{G}), \ldots, n - 3 - \lambda_n(\overline{G}).$$

Let H be any graph Q-cospectral with $G \times K_1$, then

$$q_1(H) = \frac{3n - 5 + \sqrt{(n-3)^2 + 16}}{2} > 2n - 4, \quad q_{n+1}(H) = n - 3 - \lambda_2(\overline{G}).$$

Since $q_{n+1}(H) > 0$, H has no isolated vertices. Let Δ and δ be the maximum degree and minimum degree of H, respectively. By Lemma 2.5, we have $\delta > n - 3 - \lambda_2(\overline{G}) \geqslant n - 5$, $\delta \geqslant n - 4$. Lemma 2.4 implies that $2\Delta \geqslant q_1(H) > 2n - 4$, $\Delta > n - 2$. Since H has $n + 1$ vertices, $\Delta = n - 1$ or n. If $\Delta = n$, by Lemma 3.2, we have $H = G \times K_1$. So we only need to consider the case $\Delta = n - 1$. Let a_i be the number of vertices of degree $n - i$ in H. By Lemma 2.2, we have

$$\sum_{i=1}^{4} a_i = n + 1, \quad \sum_{i=1}^{4} (n-i)a_i = n + n(n-2), \quad \sum_{i=1}^{4} (n-i)^2 a_i = n^2 + n(n-2)^2.$$

Solving the above equations, we get $a_1 = 3 - a_4$, $a_2 = n - 3 + 3a_4$, $a_3 = 1 - 3a_4$. By $a_3 \geqslant 0$, we have $a_4 = 0$, $a_1 = 3$, $a_2 = n - 3$, $a_3 = 1$. By Lemma 2.2, we have

$$6t(H) + 3(n-1)^3 + (n-3)(n-2)^3 + (n-3)^3 = 6t(G \times P_1) + n^3 + n(n-2)^3,$$

$$t(H) = t(G \times K_1) + 1.$$

Since H and $G \times K_1$ have the same number of vertices and edges, their complements $\overline{H}$ and $\overline{G} \cup K_1$ have the same number of vertices and edges. Clearly the vertex degrees of $\overline{H}$ and $\overline{G} \cup K_1$ are $1, 1, 1, 2, \ldots, 2, 3$ and $0, 2, \ldots, 2$, respectively. Then we have

$$t(H) = N_{C_3}(K_{n+1}) - (n-1)N_{K_2}(\overline{H}) + N_{K_{1,2}}(\overline{H}) - N_{C_3}(\overline{H}),$$

$$t(G \times K_1) = N_{C_3}(K_{n+1}) - (n-1)N_{K_2}(\overline{G}) + N_{K_{1,2}}(\overline{G}) - N_{C_3}(\overline{G}).$$

Considering the vertex degrees of $\overline{H}$ and $\overline{G}$, we have $N_{K_{1,2}}(\overline{H}) = 3 + n - 3 = n$, $N_{K_{1,2}}(\overline{G}) = n$. If $\overline{G}$ has no triangles, then $t(H) \leqslant t(G \times K_1)$, a contradiction to $t(H) = t(G \times K_1) + 1$. Hence $G \times K_1$ is determined by its signless Laplacian spectrum when $\overline{G}$ has no triangles.

Recall that $\overline{G}$ is a 2-regular graph. If $\overline{G}$ has a triangle, assume that $\overline{G} = C_3 \cup X$, where X is a 2-regular graph. Then $G \times K_1 = K_{1,3} \times \overline{X}$. Theorem 3.6 implies that $G \times K_1$ is not determined by its signless Laplacian spectrum if $\overline{G}$ has a triangle. $\square$

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